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On Certain Binary Functions in Operator Algebras

Ray Redheffer

University of California, Los Angeles and Mathematisches Institut I, 76128 Karlsruhe, Germany

and

Peter Volkmann

Mathematisches Institut I, 76128 Karlsruhe, Germany

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1. INTRODUCTION

As seen by [2–4], certain binary expressions

$$\lceil p, q \rceil_a, \qquad \lceil p, q \rceil_b, \qquad \lceil p, q \rceil_c, \qquad \lceil p, q \rceil_d$$

play a fundamental role in the theory of differential equations in operator algebras. Their usefulness depends on equivalences such as

$$\lceil p, q \rceil_b = \lceil p, q \rceil_d, \qquad \lceil p, q \rceil_a = \lceil p, q \rceil_c$$

that are by no means obvious. The first of these is used in [2] and is proved there. The second, which is needed in [4], is more subtle and the proof was postponed with a view to presenting it separately.

2. FOUR BINARY FUNCTIONS

Using the same notation as described in Section 1 of [4], we define the binary functions mentioned above and their counterparts with $\lfloor \ \rfloor$. The letters a, b, c, d stand for *abstract*, *ball*, *cone*, and *distance* respectively.

Definitions 1. For $p \in \partial P$ and $q \in L(X)$

$$\lceil p, q \rceil_a = \sup_{c \in \rho(p)} \operatorname{Re} cq, \qquad \lfloor p, q \rfloor_a = \inf_{c \in \rho(p)} \operatorname{Re} cq$$

where $\rho(p)$ is the set of functionals $c \in P^*$ satisfying $cI = 1$ and $\operatorname{Re} cp = 0$.

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DEFINITION 2. For $p \in \partial B$ and $q \in L(X)$

$$\ulcorner p, q \urcorner_b = \lim_{h \to 0+} \sup_{\xi \in \sigma(p,h)} \operatorname{Re}(p\xi, q\xi), \qquad \llcorner p, q \lrcorner_b = \lim_{h \to 0+} \inf_{\xi \in \sigma(p,h)} \operatorname{Re}(p\xi, q\xi)$$

where $\sigma(p, h)$ is the set of $\xi \in X$ satisfying $|\xi| = 1$ and $|p^*p\xi - \xi| < h$.

DEFINITION 3. For $p \in \partial P$ and $q \in L(X)$

$$\ulcorner p, q \urcorner_c = \lim_{h \to 0+} \sup_{\xi \in \rho(p,h)} \operatorname{Re}(\xi, q\xi), \qquad \llcorner p, q \lrcorner_c = \lim_{h \to 0+} \inf_{\xi \in \rho(\xi,h)} \operatorname{Re}(\xi, q\xi)$$

where $\rho(p, h)$ is the set of $\xi \in X$ satisfying $|(p + p^*)\xi| \leqslant h$ and $|\xi| = 1$.

DEFINITION 4. For $p \in \partial B$ and $q \in L(X)$

$$\ulcorner p, q \urcorner_d = \lim_{h \to 0+} \frac{\|p + hq\| - 1}{h}, \qquad \llcorner p, q \lrcorner_d = \lim_{h \to 0-} \frac{\|p + hq\| - 1}{h}.$$

In [2] we showed that the set $\sigma(p, h)$ is nonempty for $h > 0$ and that

$$\ulcorner p, q \urcorner_b = \ulcorner p, q \urcorner_d, \qquad \llcorner p, q \lrcorner_b = \llcorner p, q \lrcorner_d$$

for all $p \in \partial B$ and $q \in L(X)$. It is a principal objective now to prove the following:

THEOREM 1. *For all $p \in \partial P$ and $q \in L(X)$, the sets $\rho(p, h)$ and $\rho(p)$ are nonempty and*

$$\ulcorner p, q \urcorner_c = \ulcorner p, q \urcorner_a, \qquad \llcorner p, q \lrcorner_c = \llcorner p, q \lrcorner_a.$$

Since $\ulcorner p, q \urcorner_a = -\llcorner p, -q \lrcorner_a$ and $\ulcorner p, q \urcorner_c = -\llcorner p, -q \lrcorner_c$, the first equality follows from the second. Proof of the second is given next.

3. PROOF, FIRST STEP

First we show that the two sets mentioned in the theorem are nonempty.

LEMMA 1. *If $p \geqslant 0$, $|\xi| = 1$ and $\operatorname{Re}(\xi, p\xi) \leqslant \varepsilon$, then $|(p + p^*)\xi|^2 \leqslant 4\, \|p\|\, \varepsilon$.*

Proof. Let $\eta = (p + p^*)\xi$ to get

$$0 \leqslant \operatorname{Re}(\xi + \eta t, p(\xi + \eta t)) = A + Bt + Ct^2$$

where $t \in R$ and A, B, C are defined by the equation. Clearly

$$A = \operatorname{Re}(\xi, p\xi), \qquad B = |\eta|^2, \qquad C = \operatorname{Re}(\eta, p\eta)$$

and hence $0 \leqslant A \leqslant \varepsilon$, $0 \leqslant C \leqslant \|p\|\,|\eta|^2$. The inequality $B^2 \leqslant 4AC$ gives the lemma.

If $p \in \partial P$ then $p \geqslant 0$ but it is not the case that $p > 0$. The first of these statements shows that $\operatorname{Re}(p\xi, \xi) \geqslant 0$ for all $\xi \in X$ and the second that

$$\inf_{|\xi|=1} \operatorname{Re}(\xi, p\xi) = 0.$$

The lemma now shows that $\rho(p, h)$ is nonempty for $h > 0$.

To see that $\rho(p)$ is also nonempty, we use the implication

$$c \in P^* \Rightarrow cI = \|c\|,$$

which is proved by a short calculation in [4]. This allows us to state a familiar separation theorem [1] in a form suitable for our purposes:

LEMMA 2. *If K is a convex subset of $L(X)$ with $P^0 \cap K = \phi$, there exists a functional $c \in P^*$ such that*

$$\|c\| = cI = 1, \quad \operatorname{Re} cx \leqslant 0 \qquad \textit{for all} \quad x \in K.$$

The special case $K = \{p\}$ with $p \in \partial P$ gives $\operatorname{Re} cp \geqslant 0$, since $p \in P$ and $c \in P^*$. Also $\operatorname{Re} cp \leqslant 0$ by the separation theorem. Hence $\rho(p)$ is not empty.

4. PROOF, CONTINUED

We will need the following:

LEMMA 3. *Let $p \in \partial P$, $q \in L(X)$ and $\lambda \in R$. Then*

$$\inf_{|\xi|=1} \operatorname{Re}(\xi, (q + \lambda p)\xi) \leqslant \lfloor p, q \rfloor_a.$$

Proof. Let $c \in \rho(p)$, that is, $c \in P^*$, $cI = 1$, $\operatorname{Re} cp = 0$. Then the first equation below holds, and we define δ by the second:

$$\operatorname{Re} c(q + \lambda p) = \operatorname{Re} cq, \qquad \delta = \inf_{|\xi|=1} \operatorname{Re}(\xi, (q + \lambda p)\xi).$$

Clearly

$$\inf_{|\xi|=1} \operatorname{Re}(\xi, (q + \lambda p - \delta I)\xi \geqslant 0$$

and hence $q + \lambda p \geqslant \delta I$. The hypothesis $c \in P^*$ now gives

$$0 \leqslant \operatorname{Re} c(q + \lambda p - \delta I) = \operatorname{Re} cq - \delta,$$

which is to say, $\delta \leqslant \operatorname{Re} cq$. Since this holds for every $c \in \rho(p)$, the lemma follows.

We will now show that

$$\llcorner p, q \lrcorner_c \leqslant \llcorner p, q \lrcorner_a, \qquad p \in \partial P, \quad q \in L(X). \tag{1}$$

To this end let $\alpha > \llcorner p, q \lrcorner_a$ and $h > 0$. Lemma 3 shows that for each positive integer $\lambda = n$ there exists $\xi_n \in X$, $|\xi_n| = 1$ such that

$$\operatorname{Re}(\xi_n, q\xi_n) + n \operatorname{Re}(\xi_n, p\xi_n) \leqslant \alpha.$$

Since $p \geqslant 0$ this gives

$$\operatorname{Re}(\xi_n, q\xi_n) \leqslant \alpha, \qquad n \operatorname{Re}(\xi_n, p\xi_n) \leqslant \alpha + \|q\|.$$

Lemma 1 shows that $|(p + p^*)\xi_n|$ has the order of magnitude $O(1/\sqrt{n})$ and hence $\xi_n \in \rho(p, h)$ for n sufficiently large. From this follows

$$\inf_{\xi \in \rho(p, h)} \operatorname{Re}(\xi, q\xi) \leqslant \alpha.$$

First letting $h \to 0+$ and then letting $\alpha \to \llcorner p, q \lrcorner_a$, we obtain (1).

5. PROOF, CONCLUDED

We complete the proof of Theorem 1 by showing that

$$\llcorner p, q \lrcorner_a \leqslant \llcorner p, q \lrcorner_c, \qquad p \in \partial P, \quad q \in L(X). \tag{2}$$

To this end let $\beta > \llcorner p, q \lrcorner_c$, so that

$$\inf_{\xi \in \rho(p, h)} \operatorname{Re}(\xi, q\xi) < \beta, \qquad h > 0.$$

Taking $h = 1/n$, where n is a positive integer, we get $\xi_n \in X$ such that

$$\operatorname{Re}(\xi_n, q\xi_n) < \beta, \qquad |(p + p^*)\xi_n| \leqslant \frac{1}{n}, \quad |\xi_n| = 1.$$

Let us now define a nonlinear functional N: $L(X) \to R$ by the equation

$$Nz = \limsup_{n \to \infty} \operatorname{Re}(\xi_n, z\xi_n), \qquad z \in L(X).$$

Then $Nq \leqslant \beta$ and $N\lambda I = \lambda$ for $\lambda \in R^+$. The equation

$$0 \leqslant 2\mathrm{Re}(\xi_n, p\xi_n) \leqslant |(p+p^*)\xi_n| \leqslant \frac{1}{n}$$

gives $Np = 0$. For $x, y, z \in L(X)$ and $\lambda \in R^+$ it is readily checked that

$$N(x+y) \leqslant Nx + Ny, \qquad N\lambda z = \lambda Nz.$$

Hence the set K defined by

$$K = \{z \in L(X) : Nz \leqslant 0\}$$

is a convex cone. We have $P^0 \cap K = \phi$ because

$$z \in P^0 \Rightarrow \inf_{|\xi|=1} \mathrm{Re}(\xi, z\xi) > 0 \Rightarrow \inf_n \mathrm{Re}(\xi_n, z\xi_n) > 0.$$

Hence $Nz > 0$, which shows that z is not in K.

The separation theorem in Lemma 2 gives a functional $c \in P^*$ satisfying

$$cI = 1, \qquad \mathrm{Re}\, cz \leqslant 0, \qquad z \in K.$$

Since $p \in P \cap K$ we must have $\mathrm{Re}\, cp = 0$, hence $c \in \rho(p)$. From

$$N(q - \beta I) \leqslant Nq + N(-\beta I) \leqslant \beta - \beta = 0$$

follows, in succession,

$$q - \beta I \in K, \qquad \mathrm{Re}\, c(q - \beta I) \leqslant 0, \qquad \mathrm{Re}\, cq \leqslant \beta.$$

Since $c \in \rho(p)$ this gives $\lfloor p, q \rfloor_a \leqslant \beta$, and letting $\beta \to \lfloor p, q \rfloor_c$ we get (2). This completes the proof of Theorem 1.

6. A REFINEMENT

The foregoing analysis gives the following chain of inequalities, where $p \in \partial P$, $q \in L(X)$ and Z denotes the set of integers:

$$\begin{aligned} \lfloor p, q \rfloor_c &\leqslant \sup_{n \in Z} \inf_{|\xi|=1} \mathrm{Re}(\xi, (q+np)\xi) \\ &\leqslant \sup_{\lambda \in R} \inf_{|\xi|=1} \mathrm{Re}(\xi, (q+\lambda p)\xi) \leqslant \lfloor p, q \rfloor_a. \end{aligned}$$

The first inequality is implicit in the analysis of Section 4, the second is obvious, and the third follows from Lemma 3. The function

$$\phi(\lambda) = \inf_{|\xi| = 1} \operatorname{Re}(\xi, (q + \lambda p)\xi)$$

is increasing in λ because $p \geqslant 0$ and is bounded by q because p is not > 0. Combining this observation with the inequalities above we get the following refinement of Theorem 1:

THEOREM 2. *For $p \in \partial P$ and $q \in L(X)$*

$$\lfloor p, q \rfloor_a = \lfloor p, q \rfloor_c = \lim_{\lambda \to \infty} \inf_{|\xi| = 1} \operatorname{Re}(\xi, (q + \lambda p)\xi).$$

7. THE CAYLEY TRANSFORM

The Cayley transform is defined by

$$y = (x - I)(x + I)^{-1}, \qquad x = (I + y)(I - y)^{-1}.$$

As is well known, this gives a mapping $P \leftrightarrow B$ which takes the interior to the interior. It also takes the boundary to the boundary provided attention is restricted to values x, y for which $x + I$ and $y - I$ are invertible. In reviewing this matter, we will show that the sets

$$P_0 = \{x\colon \operatorname{Re}(\xi, x\xi) > 0\} \qquad \text{for} \quad \xi \neq 0$$

and

$$B_0 = \{y\colon \operatorname{Re}(y\eta, y\eta) < 1\} \qquad \text{for} \quad |\eta| = 1$$

are mapped onto each other. Though the proof is in some respects simpler than those usually given, the discussion should be regarded as expository.

Suppose, then, that $x \in P_0$ and $\eta = y\xi$. We want to show

$$|\eta| < |\xi|, \qquad \xi \neq 0.$$

From $\eta = y\xi$ follows $(x + I)\eta = (x - I)\xi$, hence

$$\xi + \eta = x(\xi - \eta). \tag{3}$$

For $\xi \neq \eta$ the hypothesis $x \in P_0$ gives

$$|\xi|^2 - |\eta|^2 = \operatorname{Re}(\xi - \eta, \xi + \eta) = \operatorname{Re}(\xi - \eta, x(\xi - \eta)) > 0.$$

If $\xi = \eta$ then $\xi = 0$ by (3).

Suppose next that $y \in B_0$ and $\eta = x\xi$. We want to show that

$$\operatorname{Re}(\xi, \eta) > 0, \qquad \xi \neq 0.$$

From $\eta = x\xi$ follows $(I - y)\eta = (I + y)\xi$, hence

$$\eta - \xi = y(\eta + \xi). \tag{4}$$

For $\eta + \xi \neq 0$ the hypothesis $y \in B_0$ gives

$$|\eta - \xi|^2 < |\eta + \xi|^2$$

so $\operatorname{Re}(\xi, \eta) > 0$. If $\eta + \xi = 0$ then $\xi = 0$ by (4). This completes the proof.

One of the main uses of the binary expressions studied here is in the theory of operator differential equations such as

$$u' = f(t, u), \qquad v' = g(t, v)$$

where u and v are functions $R \to L(X)$ and the prime denotes the derivative in some suitable sense. The main property we need is the Leibniz rule

$$(uv)' = uv' + u'v.$$

The sets B or P are *positively invariant* if

$$u(t_0) \in B \Rightarrow u(t) \in B \qquad \text{or} \quad v(t_0) \in P \Rightarrow v(t) \in P, \qquad t > t_0$$

respectively. Aside from smoothness conditions on f and g that will not be discussed here, the decisive requirement is a version of the tangent condition of Nagumo. In view of the equation

$$\lfloor p, q \rfloor_b = \lfloor p, q \rfloor_d$$

the tangent condition for B is equivalent to

$$u \in \partial B \Rightarrow \lfloor u, u' \rfloor_b \leqslant 0.$$

It is less evident, but true, that the tangent condition for the cone is equivalent to

$$v \in \partial P \Rightarrow \lceil v, v' \rceil_c \geqslant 0.$$

Historically, theorems of invariance for B and P come from quite different traditions and are associated with distinctly different problems. It is therefore of interest to know whether these theorems can be deduced one from the other via a Cayley transform. A first step in that direction is given by:

THEOREM 3. *Suppose the differentiable functions u and v are related by a Cayley transform, so that*

$$u=(v-I)(v+I)^{-1}, \qquad v=(I+u)(I-u)^{-1}. \tag{5a, 5b}$$

Then $\lfloor u, u' \rfloor_b \leqslant 0 \Leftrightarrow \lceil v, v' \rceil_c \geqslant 0$ *for* $u \in \partial B$ *and* $v \in \partial P$.

It is implied by the hypothesis, and now assumed explicitly, that $I-u$ and $I+v$ are both invertible at the value t in question.

8. PROOF OF THEOREM 3

Let us begin by obtaining a relation between u' and v'. The first equation below follows from (5b) and the other two equations follow from the first:

$$v-vu=I+u, \qquad v'-vu'-v'u=u', \qquad v'(I-u)=(I+v)u'.$$

By (5b) again

$$I+v=(I-u+I+u)(I-u)^{-1}=2(I-u)^{-1}.$$

These results together give the equation

$$v'=2(I-u)^{-1}\,u'(I-u)^{-1}. \tag{6}$$

Although not needed here, it is worth mentioning that a similar calculation gives

$$u'=2(I+v)^{-1}\,v'(I+v)^{-1}.$$

We set $|\eta|=1$ and define

$$\lambda\xi=(I-u)\eta, \qquad \lambda=|(I-u)\eta|. \tag{7}$$

Since $I-u$ is invertible, $\lambda \neq 0$ and hence $|\xi|=1$. The fact that $\lambda \neq 0$ is used again below.

The main equation underlying the following analysis is

$$\lambda(I-u^*)(v+v^*)\xi=2(I-u^*u)\eta. \tag{8}$$

This is obtained when we eliminate the factor $v+v^*$ on the left by use of

$$v(I-u)=I+u, \qquad (I-u^*)v^*=I+u^*.$$

To make the argument easier to follow, we will replace $\rho(p, h)$ and $\sigma(p, h)$ by simpler sets $\rho(p)$, $\sigma(p)$ corresponding to $h=0$. Thus, for the moment,

$$\ulcorner p, q \urcorner_c = \sup_{\xi \in \rho(p)} \operatorname{Re}(\xi, q\xi), \qquad \llcorner p, q \lrcorner_b = \inf_{\eta \in \sigma(p)} \operatorname{Re}(p\eta\, q\eta),$$

where $\rho(p)$ and $\sigma(p)$ are sets of vectors in X defined by

$$|\xi| = 1, \qquad (p+p^*)\xi = 0, \qquad \|\eta| = 1, \quad p^*p\eta = \eta$$

respectively. Since (8) does not depend on this simplification, it will be found that only minor changes are needed for the general case.

Equation (8) together with (7) shows that the conditions $\xi \in \rho(v)$ and $\eta \in \sigma(u)$ are equivalent. From (7) and (6) we get

$$\lambda^2 \operatorname{Re}(\xi, v'\xi) = 2 \operatorname{Re}((I-u)\eta, (I-u)^{-1} u'\eta).$$

Since $(I-u)\eta = -(I-u^*)\, u\eta$ when $u^*u\eta = \eta$, the result simplifies to

$$\lambda^2 \operatorname{Re}(\xi, v'\xi) = -2 \operatorname{Re}(u\eta, u'\eta).$$

This follows from (7) with $\eta \in \sigma(u)$. Thus the two conditions

$$\operatorname{Re}(\xi, v'\xi) \geqslant 0, \qquad \operatorname{Re}(u\eta, u'\eta) \leqslant 0 \tag{9a, 9b}$$

are equivalent in the following sense: If (9a) holds for at least one $\xi \in \rho(v)$, then (9b) holds for at least one $\eta \in \sigma(u)$. If (9a) holds for all $\xi \in \rho(v)$, then (9b) holds for all $\eta \in \sigma(u)$. This gives Theorem 3 in the simplified case.

In the general case, Equation (8) shows that the two expressions

$$|(v+v^*)\xi|, \qquad |(I-u^*u)\eta|$$

have the same order of magnitude; note that $|\eta| = 1$ gives both an upper and a lower bound for λ, since $I-u$ is invertible. If we denote either of these expressions by h the general definition introduces an error term ε_h which tends to 0, and the argument is completed much as before.

REFERENCES

1. N. Dunford and J. T. Schwartz, "Linear Operators Part I: General Theory," p. 417, Interscience, New York, 1957.
2. R. Redheffer and P. Volkmann, Differential equations in operator algebras I, *Math. Ann.* **239** (1979), 97–110.
3. R. Redheffer and P. Volkmann, Differential equations and divisibility in operator algebras, *J. Differential Equations* **78** (1989), 136–143.
4. R. Redheffer and P. Volkmann, Differential equations in operator algebras, II. Invariance of the order cone, *J. Differential Equations* **130** (1996), 356–367.